

ON THE DYNAMICS OF RATIONAL MAPS WITH TWO FREE CRITICAL POINTS

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ABSTRACT. In this paper we discuss the dynamical structure of the rational family (f_t) given by

$$f_t(z) = tz^m \left(\frac{1-z}{1+z} \right)^n \quad (m \geq 2, t \neq 0).$$

Each map f_t has two super-attracting immediate basins and two free critical points. We prove that for $0 < |t| \leq 1$ and $|t| \geq 1$, either of these basins is completely invariant and at least one of the free critical points is inactive. Based on this separation we draw a detailed picture the structure of the dynamical and the parameter plane.

1. INTRODUCTION

Non-trivial rational families (f_t) normally contain specific maps of different character with most interesting and unexpected Julia sets:

- totally disconnected Julia sets (Cantor sets) occur in any family $z \mapsto z^d + t$;
- Julia sets consisting of uncountably many (a Cantor set of) quasi-circles occur in the McMullen family $z \mapsto z^m + t/z^n$, which was introduced in [9]. The number of papers on various features of this family is legion; [3] marks the preliminary end of a long list of papers.
- Julia sets that are Sierpiński curves (Tan Lei and J. Milnor [12] were the first to construct examples with this property) again in the McMullen family [16], the Morosawa-Pilgrim family $z \mapsto t(1 + \frac{(4/27)z^3}{1-z})$ [4, 18], and the family $t \mapsto -\frac{t(z^2-2)^2}{4z^2-1}$ [7].
- In any reasonable family, copies of the Mandelbrot sets of the families $z \mapsto z^d + t$ are dense in the bifurcation locus – the Mandelbrot set is universal [10].

Each of these families has just one *free* critical point (or several free critical points which have the same dynamical behaviour, this happens, for example, in the McMullen family; the quasi-conjugate family

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$F_t(z) = z^m(1 + t/z)^d$ has just one free critical point). In contrast to this the rational maps

$$(1) \quad f_t(z) = tz^m \left(\frac{1-z}{1+z} \right)^n \quad (m \geq 2, n \in \mathbb{N}, d = m + n, t \neq 0)$$

in the family under consideration have two free critical points. In this paper we will give a complete description of the parameter plane and the various dynamical planes. For basic notations and results the reader is referred to the texts [1, 2, 11, 9, 15].

2. NOTATION

The rational map (1) has

- two super-attracting fixed points 0 and ∞ with corresponding basins \mathcal{A}_t and \mathcal{B}_t , respectively. Then either \mathcal{A}_t , say, is completely invariant or else has a single pre-image \mathcal{A}_t^* that is mapped in a $(n : 1)$ -manner onto \mathcal{A}_t , which will be written as

$$\mathcal{A}_t^* \xrightarrow{n:1} \mathcal{A}_t;$$

- two free critical points

$$\alpha = -\frac{n}{m} + \sqrt{1 + \left(\frac{n}{m}\right)^2} \quad \text{and} \quad \beta = -\frac{n}{m} - \sqrt{1 + \left(\frac{n}{m}\right)^2}$$

and critical values

$$v_t^\alpha = f_t(\alpha) = tv_1^\alpha \quad \text{and} \quad v_t^\beta = f_t(\beta) = tv_1^\beta;$$

- two *escape loci* Ω^α and Ω^β , with $t \in \Omega^\alpha$ and $t \in \Omega^\beta$ if and only if $f_t^k(\alpha) \rightarrow 0$ and $f_t^k(\beta) \rightarrow \infty$, respectively, as $k \rightarrow \infty$;
- two *residual sets* $\Omega_{\text{res}}^\alpha$ and $\Omega_{\text{res}}^\beta$, with $t \in \Omega_{\text{res}}^\alpha$ and $t \in \Omega_{\text{res}}^\beta$ if and only if $v_t^\beta \in \mathcal{A}_t$ and $v_t^\alpha \in \mathcal{B}_t$, respectively.

The notation of the residual sets indicates that $\Omega_{\text{res}}^\alpha$ is related to Ω^α rather than Ω^β . The open sets Ω^α and Ω^β are in a natural way subdivided into

- Ω_0^α resp. Ω_0^β : $v_t^\alpha \in \mathcal{A}_t$ resp. $v_t^\beta \in \mathcal{B}_t$, and
- Ω_k^α resp. Ω_k^β : $f_t^k(v_t^\alpha) \in \mathcal{A}_t$, but $f_t^{k-1}(v_t^\alpha) \notin \mathcal{A}_t$ resp.
 $f_t^k(v_t^\beta) \in \mathcal{B}_t$, but $f_t^{k-1}(v_t^\beta) \notin \mathcal{B}_t$ ($k \geq 1$).

Hitherto, f_t is hyperbolic and the Fatou set of f_t consists of the basins \mathcal{A}_t and \mathcal{B}_t , and their pre-images, if any. However, there may and will be also other hyperbolic components. By \mathbf{W}_k^α and \mathbf{W}_k^β we denote the open sets such that α and β belongs to some (super-)attracting cycle of Fatou domains U_1, \dots, U_k , respectively, not containing 0 and ∞ .

The *bifurcation* locus \mathbf{B} of the family $(f_t)_{0 < |t| < \infty}$ is the set of t such that the Julia set \mathcal{J}_t does not move continuously over any neighbourhood of t , see McMullen [9]. In order that $t \in \mathbf{B}$ it is necessary and sufficient that at least one of the free critical points is *active*. Thus $\mathbf{B} = \mathbf{B}^\alpha \cup \mathbf{B}^\beta$, where $t \in \mathbf{B}^\alpha$ resp. $t \in \mathbf{B}^\beta$ means that α resp. β is active. It is *a priori* not excluded that \mathbf{B}^α and \mathbf{B}^β overlap. Although there is just one parameter plane, each point of this plane carries at least two pieces of information, so one could also speak of the v_t^α - and v_t^β -plane.

We also set

$$Q_0(t) = v_t^\alpha = tv_1^\alpha \quad \text{and} \quad Q_k(t) = f_t^k(v_t^\alpha) = f_t(Q_{k-1}(t)) \quad (k \geq 1)$$

and note that Q_k is a rational function of degree $1 + d + \dots + d^k = \frac{d^{k+1}-1}{d-1}$ with a zero of order $\frac{m^{k+1}-1}{m-1}$ at the origin.

From

$$-1/f_t(-1/z) = f_{(-1)^{d+1}/t}(z) \quad (d = m + n)$$

it follows that f_t is conjugate to $f_{1/t}$ if d is odd, and to $f_{-1/t}$ if d is even, hence $\Omega^\alpha = \pm 1/\Omega^\beta$, and this is also true for Ω_k^α and Ω_k^β , $\Omega_{\text{res}}^\alpha$ and $\Omega_{\text{res}}^\beta$, \mathbf{W}_k^α and \mathbf{W}_k^β , and \mathbf{B}^α and \mathbf{B}^β . This also indicates that the circle $|t| = 1$ plays a distinguished role with strong impact on what follows.

LEMMA 1. *For every $m \geq 2$, $n \geq 1$ there exists some $r > 0$, such that for $0 < |t| \leq 1$ the disc $\Delta_{r|t|} : |z| < r|t|$ contains $f_t(\overline{\Delta}_{r|t|} \cup [0, 1])$, but does not contain v_t^β .*

Proof. We will first consider f_1 and show that there exists some disc $\Delta_r : |z| < r$ such that $f_1(\overline{\Delta}_r \cup [0, 1]) \subset \Delta_r$ holds. This is easy to show if $n < m$ for $r = \frac{1}{3}$:

$$|f_1(z)| \leq 3^{-m} 2^n < \frac{1}{3}$$

holds if $|z| \leq \frac{1}{3}$ and $m > n \geq 1$, and from

$$0 \leq f_1(x) \leq x^2 \frac{1-x}{1+x} \leq \frac{1}{2}(5\sqrt{5} - 11) < \frac{1}{10} \quad (0 \leq x \leq 1)$$

the assertion follows.

We now consider the case $n \geq m$. Then $f_1(\overline{\Delta}_r) \subset \Delta_r$ holds as long as

$$g(r) = r^{m-1} \left(\frac{1+r}{1-r} \right)^n < 1,$$

and f_1 maps $[0, 1]$ into Δ_r provided

$$v_1^\alpha = \max_{0 \leq x \leq 1} x^m \left(\frac{1-x}{1+x} \right)^n < r.$$

Since g is increasing this may be achieved if $g(v_1^\alpha) < 1$ holds. To prove this we note that $\sqrt{1+\tau}-1 = \frac{\tau}{2\sqrt{1+\theta\tau}}$ ($0 < \theta < 1$, $\tau = \frac{m^2}{n^2} \leq 1$) implies $\frac{m}{2\sqrt{2n}} < \alpha < \frac{m}{2n}$, while from $\log \frac{1-x}{1+x} < -2x$ ($0 < x < 1$) it follows that

$$v_1^\alpha < \left(\frac{m}{2n}\right)^m e^{-2\frac{m}{2\sqrt{2}}} = \left(\frac{m}{2e^{\frac{1}{\sqrt{2}}n}}\right)^m < \left(\frac{m}{4n}\right)^m = \mu^m.$$

Moreover, from

$$\log \frac{1+x}{1-x} = 2x \left(1 + \frac{1}{3}x^2 + \frac{1}{5}x^4 + \dots\right) \leq 2x \left(1 + \frac{x^2}{3} \frac{1}{1-x^2}\right) \leq 2x \left(1 + \frac{1}{45}\right),$$

which holds for $x = \left(\frac{m}{4n}\right)^{m-1} \leq \frac{1}{4}$, we obtain

$$\left(\frac{1+\mu^m}{1-\mu^m}\right)^n = \left(\frac{1+\frac{m}{4}\frac{\mu^{m-1}}{n}}{1-\frac{m}{4}\frac{\mu^{m-1}}{n}}\right)^n \leq e^{\frac{23}{45}m\mu^{m-1}} < \left(e^{\left(\frac{m}{4n}\right)^{m-1}}\right)^m.$$

Thus $g(v_1^\alpha) < 1$ follows from $\left(\frac{m}{4n}\right)^{m-1} e^{\left(\frac{m}{4n}\right)^{m-1}} \leq \frac{1}{4}e^{\frac{1}{4}} < 1$.

With this choice of $r \in (0, 1)$ it is clear that v_t^β belongs to Δ_r if $|t|$ is small. For individual $0 < |t| \leq 1$, $f_t(z) = tf_1(z)$ maps $\overline{\Delta_{r|t|}} \cup [0, 1]$ into $\Delta_{r|t|}$, while $v_t^\beta \notin \Delta_{r|t|}$ follows from $|v_t^\beta| = |t|/v_1^\alpha > |t| > r|t|$. \square

3. THE ESCAPE LOCI

The purpose of Lemma 1 is twofold. First of all it shows that the critical points α and β cannot be simultaneously active, and the bifurcation sets \mathbf{B}^α and \mathbf{B}^β are separated by the unit circle $|t| = 1$. Secondly, the condition $v_t^\beta \notin \Delta_{r|t|}$ ($0 < |t| \leq 1$) ensures that in an exhaustion (D_κ) of \mathcal{A}_t starting with $D_0 = \Delta_{r|t|}$, D_κ is simply connected as long as $\beta \notin D_\kappa$, and $f_t : D_\kappa \xrightarrow{d:1} D_{\kappa-1}$ has degree $d = m + n$. In particular, for $t \in \Omega_{\text{res}}^\alpha$ there exists some simply connected and forward invariant domain $D_\kappa \subset \mathcal{A}_t$ that contains v_t^β .

We note some more simple consequences of Lemma 1; our focus is on the critical point α and the α -sets.

- $\{t : 0 < |t| \leq 1\} \subset \Omega_0^\alpha$;
- $\overline{\Omega_{\text{res}}^\alpha} \subset \mathbb{D}$;
- α is inactive on $0 < |t| \leq 1$;
- $\bigcup_{k \geq 1} (\Omega_k^\alpha \cup \mathbf{W}_k^\alpha) \subset \{t : 1 < |t| < T\}$ for some $T = T_{mn} > 1$;
- $\mathbf{B}^\alpha \subset \{t : 1 < |t| < T\}$ for some $T = T_{mn} > 1$.

The consequences for the dynamical planes are as follows.

THEOREM 1. *For $t \in \Omega_0^\alpha$, the basin \mathcal{A}_t is completely invariant, and any other Fatou component is simply connected. Moreover,*

- for $t \in \Omega_0^\alpha \cap \Omega_0^\beta$ also \mathcal{B}_t is completely invariant, the Julia set $\mathcal{J}_t = \partial\mathcal{A}_t = \partial\mathcal{B}_t$ is a quasi-circle, and f_t is quasi-conformally conjugate to $z \mapsto z^d$;
- for $t \in \Omega_{\text{res}}^\alpha$, \mathcal{A}_t is infinitely connected and the Fatou set consists of \mathcal{A}_t , \mathcal{B}_t , and the predecessors of \mathcal{B}_t of any order.

Proof. To prove complete invariance of \mathcal{A}_t we first assume $0 < |t| \leq 1$. Then \mathcal{A}_t contains the interval $[0, 1]$ by Lemma 1, hence is completely invariant. If, however, $|t| > 1$, then \mathcal{B}_t is completely invariant, and any other Fatou component is simply connected. Assuming $1 \notin \mathcal{A}_t$ ($t \in \Omega_0^\alpha$, $|t| > 1$) we obtain either $f_t : \mathcal{A}_t^* \xrightarrow{n:1} \mathcal{A}_t$ with $n = (n-1) + 1$ critical points if $\alpha \in \mathcal{A}_t^*$ or else $f_t : \mathcal{A}_t \xrightarrow{m:1} \mathcal{A}_t$ with $m = (m-1) + 1$ critical points if $\alpha \in \mathcal{A}_t$, this contradicting simple connectivity of both domains \mathcal{A}_t and \mathcal{A}_t^* by the Riemann-Hurwitz formula.

The first assertion is obvious since \mathcal{B}_t shares the properties of \mathcal{A}_t and f_t is hyperbolic.

The second assertion follows from the Riemann-Hurwitz formula, since $f_t : \mathcal{A}_t \xrightarrow{d:1} \mathcal{A}_t$ has degree d and $r = (m-1) + (n-1) + 1 + 1 = d$ critical points $0, 1$ (if $n > 1$), α , and β . \square

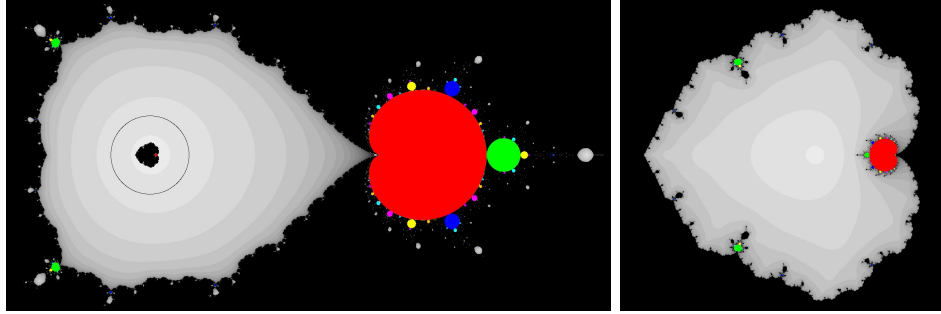


FIGURE 1-left: The α -parameter plane for $f_t(z) = tz^2 \frac{1-z}{1+z}$ displaying the unit circle, Ω^α in gray, $\Omega_{\text{res}}^\alpha$ and $\Omega_{\text{res}}^\beta$ in black (in- and outside the unit circle), and \mathbf{W}^α (coloured). FIGURE 1-right: A neighbourhood of the origin displaying $\Omega_{\text{res}}^\alpha$ surrounded by points of Ω_0^α (black), Ω_k^β ($k \geq 1$, black, small), and \mathbf{W}^β (coloured).

THEOREM 2. $\Omega_0^\alpha \cup \{0\}$, $\Omega_{\text{res}}^\alpha \cup \{0\}$, and the connected components of Ω_k^α ($k \geq 1$) are simply connected domains. Riemann maps onto \mathbb{D} are given by any branch of $\sqrt[n]{E_0(t)}$, $\sqrt[n]{E_{\text{res}}(t)}$, and $\sqrt[n]{E_k(t)}$, respectively.

For the proof we need two auxiliary results on the maps

$$\begin{aligned}
 E_0(t) &= t(\Phi_t(v_t^\alpha))^{m-1} & (t \in \Omega_0^\alpha), \\
 E_{\text{res}}(t) &= t(\Phi_t(v_t^\beta))^{m-1} & (t \in \Omega_{\text{res}}^\alpha), \text{ and} \\
 E_k(t) &= t^{\frac{1}{m-1}} \Phi_t(f^k(v_t^\alpha)) & (t \in \Omega_k^\alpha, k \geq 1),
 \end{aligned}
 \tag{2}$$

where Φ_t denotes the Böttcher function to the fixed point $z = 0$. In the first step (Lemma 2) of the proof of Theorem 2 we will show that the functions (2) provide proper maps on $\mathbb{D} \setminus \{0\}$ and \mathbb{D} , respectively, which are only ramified over the origin. In the second step (Lemma 3) this will be used to show that the corresponding domains (with 0 included, if necessary) are simply connected.

The solution to Böttcher's functional equation

$$(3) \quad \Phi_t(f_t(z)) = t\Phi_t(z)^m \quad (\Phi_t(z) \sim z \text{ as } z \rightarrow 0)$$

is locally given by

$$\Phi_t(z) = \lim_{k \rightarrow \infty} \sqrt[m^k]{f_t^k(z)/t^{1+m+\dots+m^{k-1}}} = t^{-\frac{1}{m-1}} \lim_{k \rightarrow \infty} \sqrt[m^k]{f_t^k(z)};$$

it conjugates f_t to $\zeta \mapsto \zeta^m$. This conjugation holds throughout \mathcal{A}_t in the third case, when Φ_t maps \mathcal{A}_t conformally onto the disc $|z| < |t|^{-\frac{1}{m-1}}$; the maps E_k are analytic and well-defined on the components of Ω_k^α , $k \geq 1$.

In the first case the conjugation holds on some simply connected neighbourhood of $z = 0$ that contains $z = 0$ and $z = v_t^\alpha$, but does not contain $z = 1$. The analytic continuation of Φ_t causes singularities at $z = 1$ and its preimages under f_t^k , nevertheless $|\Phi_t(z)|$ is well-defined on \mathcal{A}_t and $|\Phi_t(z)| \rightarrow |t|^{-\frac{1}{m-1}}$ as $z \rightarrow \partial\mathcal{A}_t$ holds anyway. Thus $E_0(t) = t\Phi_t(v_t^\alpha)^{m-1}$ is holomorphic on Ω_0^α and zero-free, with $E_0(t) \sim t(v_t^\alpha)^{m-1} = f_1(\alpha)^{m-1}t^m$ as $t \rightarrow 0$.

In the second case we construct an exhaustion (D_κ) of \mathcal{A}_t such that $f_t : D_\kappa \xrightarrow{d:1} D_{\kappa-1}$ has degree d and D_κ is simply connected for $\kappa \leq \kappa_0$ with $v_t^\beta \in D_{\kappa_0}$ and $\beta \in D_{\kappa_0+1} \setminus D_{\kappa_0}$. This is possible by Lemma 1, and the procedure applied to $t^{-\frac{1}{m-1}}\Phi_t(v_t^\alpha)$ on Ω_0^α also applies to $t^{-\frac{1}{m-1}}\Phi_t(v_t^\beta)$ on $\Omega_{\text{res}}^\alpha$.

LEMMA 2. *The functions in (2) are well-defined and provide proper maps from $\Omega_0^\alpha \cup \{0\}$, $\Omega_{\text{res}}^\alpha \cup \{0\}$, and the connected components of Ω_k^α with $k \geq 1$, respectively, onto the unit disc \mathbb{D} .*

Proof. To prove that $|E_0(t)| \rightarrow 1$ as $t \in \Omega_0^\alpha$ tends to $\partial\Omega_0^\alpha \setminus \{0\}$ we choose any disc $\Delta : |z| < r$ that is invariant under f_t for every $t \in \Omega_0^\alpha$. This is possible since Ω_0^α is contained in some disc $|t| < T$, hence we may choose $r < 1$ such that $Tr^{m-1}(\frac{1+r}{1-r})^n = 1$ holds. By $k = k(t)$ we denote the largest integer such that $f_t^k(v_t^\alpha) \notin \Delta$. Then $k(t) \rightarrow \infty$ as $t \rightarrow \partial\Omega_0^\alpha \setminus \{0\}$, and $|f_t^{k(t)}(v_t^\alpha)| \geq r$ implies

$$\liminf_{t \rightarrow \Omega_0^\alpha \setminus \{0\}} |\Phi_t(v_t^\alpha)| \geq \lim_{t \rightarrow \Omega_0^\alpha \setminus \{0\}} |t|^{-\frac{1}{m-1}} \sqrt[m^{k(t)}]{r} = |t|^{-\frac{1}{m-1}},$$

while $|\Phi_t(z)| < |t|^{-\frac{1}{m-1}}$ is always true. Thus E_0 maps each connected component of Ω_0^α properly onto $\mathbb{D} \setminus \{0\}$. It follows that the origin is removable for (a zero of) E_0 , and $\Omega_0^\alpha \cup \{0\}$ is a domain which is mapped by E_0 properly with degree m onto the unit disc \mathbb{D} .

If $t \in \Omega_k^\alpha$ for some $k \geq 1$, then again $|E_k(t)|$ tends to 1 as $t \rightarrow \partial\Omega$, where Ω is any component of Ω_k^α . Thus E_k is a proper map of Ω onto \mathbb{D} . We will prove that E_k is ramified only over zero even for $k \geq 0$, that is $E'_k(t) = 0$ implies $E_k(t) = 0$. This is a well-known procedure, the idea of which is due to Roesch [13], and outlined in detail for the Morosawa-Pilgrim family $z \mapsto t(1 + \frac{(4/27)z^3}{1-z})$ in [18], Lemma 2.

We take any $t_0 \in \Omega_k^\alpha$ and choose $\varepsilon > 0$ such that for t sufficiently close to t_0 , the closed disc $\Delta_{3\varepsilon} : |w - v_{t_0}^\alpha| \leq 3\varepsilon$ belongs to the Fatou component D_{t_0} of f_{t_0} containing $v_{t_0}^\alpha$ (D_{t_0} is a predecessor of \mathcal{A}_{t_0} of order $\ell \geq 0$). Furthermore let $\eta_t : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ be any diffeomorphism such that $\eta_t(w)$ depends analytically on t , $\eta_t(w) = w$ holds on $|w - v_{t_0}^\alpha| \geq 3\varepsilon$ and $\eta_t(w) = w + (v_t^\alpha - v_{t_0}^\alpha)$ on $|w - v_{t_0}^\alpha| < \varepsilon$. Then $g_t = \eta_t \circ f_{t_0} : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ is a quasi-regular map which equals f_{t_0} on $\widehat{\mathbb{C}} \setminus f_{t_0}^{-1}(\Delta_{3\varepsilon})$, and is analytic on $\widehat{\mathbb{C}} \setminus f_{t_0}^{-1}(A)$ with $A = \{w : \varepsilon \leq |w - v_{t_0}^\alpha| \leq 3\varepsilon\}$. To apply Shishikura's qc-lemma [14] we need to know that g_t is uniformly K -quasi-regular, that is, all iterates g_t^p are K -quasi-regular with one and the same K . This is obviously true if the sets $f_{t_0}^{-p}(A)$ ($p = 1, 2, \dots$) are visited at most once by any iterate of g_t . This is trivial if $k \geq 1$: the sets $f_{t_0}^{-p}(A)$ belong to different Fatou components, namely predecessors of D_{t_0} of order p . If $k = 0$ the argument is different. Let $\Delta_0 : |z| < \delta$ be such that $f_{t_0}(\overline{\Delta_0}) \subset \Delta_0$ and set $\Delta_\nu = f_{t_0}^{-1}(\Delta_{\nu-1})$. Then choosing $\epsilon > 0$ sufficiently small we have $A \subset \Delta_\ell \setminus \overline{\Delta_{\ell-1}}$ for some ℓ and $f_{t_0}^{-p}(A) \subset \Delta_{\ell+p} \setminus \overline{\Delta_{\ell+p-1}}$. By the above mentioned qc-lemma, g_t is quasi-conformally conjugate to some rational function

$$R_t = h_t \circ g_t \circ h_t^{-1}.$$

The quasi-conformal mapping $h_t : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ is uniquely determined by the normalisation $h_t(z) = z$ for $z = 0, \alpha, 1$, and depends analytically on the parameter t . Also h_t is analytic on $\widehat{\mathbb{C}} \setminus \overline{\bigcup_{p \geq 0} f_{t_0}^{-p}(A)}$, which, in particular, contains the points 0 , v_t^α , and $v_{t_0}^\alpha$. We set $z_0 = h_t(-1)$ to obtain $R_t(z) = a(t)z^m \left(\frac{1-z}{z-z_0}\right)^n$. Since $h_t(\alpha) = \alpha$, R_t has a critical point at $z = \alpha$, and solving $R'_t(\alpha) = 0$ for z_0 yields $z_0 = -1$, thus

$$R_t(z) = a(t)z^m \left(\frac{1-z}{1+z}\right)^n.$$

From $R_t = h_t \circ \eta_t \circ f_{t_0}$ and $h_t(\alpha) = \alpha$, however, it follows that

$$a(t)v_1^\alpha = R_t(\alpha) = h_t \circ \eta_t \circ f_{t_0}(\alpha) = h_t \circ \eta_t(v_{t_0}^\alpha) = h_t(v_t^\alpha),$$

hence $R_t(z) = f_\tau(z)$ with $\tau = \tau(t) = h_t(v_t^\alpha)/v_1^\alpha$ and $v_\tau^\alpha = h_t(v_t^\alpha)$; in particular, τ depends analytically on t . On some neighbourhood of $z = 0$ we have

$$\begin{aligned} (t_0/\tau)^{\frac{1}{m-1}} \Phi_{t_0} \circ h_t^{-1} \circ f_\tau &= (t_0/\tau)^{\frac{1}{m-1}} \Phi_{t_0} \circ g_t \circ h_t^{-1} \\ &= (t_0/\tau)^{\frac{1}{m-1}} \Phi_{t_0} \circ \eta_t \circ f_{t_0} \circ h_t^{-1} \\ &= (t_0/\tau)^{\frac{1}{m-1}} \Phi_{t_0} \circ f_{t_0} \circ h_t^{-1} \\ &= (t_0/\tau)^{\frac{1}{m-1}} t_0 (\Phi_{t_0} \circ h_t^{-1})^m \\ &= \tau ((t_0/\tau)^{\frac{1}{m-1}} \Phi_{t_0} \circ h_t^{-1})^m, \end{aligned}$$

hence $\phi_\tau = (t_0/\tau)^{\frac{1}{m-1}} \Phi_{t_0} \circ h_t^{-1}$ solves Böttcher's functional equation

$$\phi_\tau \circ f_\tau(z) = \tau(\phi_\tau(z))^m.$$

Since τ and h_t depend analytically on t , this is also true for h_t^{-1} , which is not self-evident. Also from $h_t(g_t(z)) = f_\tau(h_t(z)) \sim \tau h_t(z)^m$ and $g_t(z) = f_{t_0}(z) \sim t_0 z^m$ as $z \rightarrow 0$ it follows that $h_t(t_0 z^m) \sim \tau h_t(z)^m$, hence $h_t(z) \sim \sqrt[m-1]{t_0/\tau} z$, $h_t^{-1}(z) \sim \sqrt[m-1]{\tau/t_0} z$ and $\phi_\tau(z) \sim \lambda z$ as $z \rightarrow 0$, with $\lambda^{m-1} = 1$. This implies $\phi_\tau = \lambda \Phi_\tau$ by uniqueness of the Böttcher coordinate, and from $\tau(t_0) = t_0$ and analytic dependence on t it follows that $\lambda = 1$ and $\phi_\tau = \Phi_\tau$, hence

$$\begin{aligned} E_k(\tau) &= \tau^{\frac{1}{m-1}} \Phi_\tau(Q_k(\tau)) = \tau^{\frac{1}{m-1}} \Phi_\tau(f_\tau^k(v_\tau^\alpha)) \\ &= t_0^{\frac{1}{m-1}} \Phi_{t_0} \circ h_t^{-1}(f_\tau^k(v_\tau^\alpha)) = t_0^{\frac{1}{m-1}} \Phi_{t_0} \circ f_{t_0}^k \circ h_t^{-1}(v_\tau^\alpha) \\ &= t_0^{\frac{1}{m-1}} \Phi_{t_0}(f_{t_0}^k(v_t^\alpha)) \quad \text{if } k \geq 1, \text{ and} \\ E_0(\tau) &= t_0(\Phi_{t_0}(v_t^\alpha))^m. \end{aligned}$$

Since $t \mapsto \tau$ is locally univalent, E_k is univalent at t_0 if and only if the map $t \mapsto t_0^{\frac{1}{m-1}} \Phi_{t_0}(f_{t_0}^k(v_t^\alpha))$ is univalent on some neighbourhood of t_0 . If $k \geq 1$, Φ_{t_0} is univalent on \mathcal{A}_{t_0} , and $f_{t_0}^k$ is univalent on $|z - v_{t_0}^\alpha| < \delta$ provided $Q_k(t_0) = f_{t_0}^k(v_{t_0}^\alpha) \neq 0$, while $f_{t_0}^k$ is n -valent at $v_{t_0}^\alpha$ if $Q_k(t_0) = 0$. In case of $k = 0$ we note that Φ_{t_0} is locally univalent on some forward invariant domain D that contains 0 and $v_{t_0}^\alpha$, and $v_t^\alpha = t v_1^\alpha \neq 0$ is trivially univalent. \square

The proof of Theorem 2 will be finished by

LEMMA 3. *Let h be a proper map of degree m of the domain D onto the unit disc \mathbb{D} , and assume that h is ramified exactly over zero, that is, $h'(z) = 0$ implies $h(z) = 0$. Then D is simply connected and h has a single zero on D .*

Proof. Assume that h has zeros with multiplicities m_ν ($1 \leq \nu \leq n$). Then h has degree $d = m_1 + \dots + m_n$ and $r = d - n$ critical points. The Riemann-Hurwitz formula then yields $\#D - 2 = -d + r = -n$, hence $\#D = 2 - n$, which only is possible if $n = 1$ and $\#D = 1$. \square

REMARK. Each connected component of Ω_k^α contains a zero of $Q_k(t) = tf_1(Q_{k-1}(t))$ which is not a zero of Q_{k-1} , hence is a zero of $Q_{k-1}(t) - 1$. Thus Ω_k^α consists of at most $\frac{d^k-1}{d-1}$ connected components.

4. THE HYPERBOLIC LOCI

The bifurcation locus \mathbf{B}^β is contained in some annulus $\delta < |t| < 1$, and this also holds for \mathbf{W}^β . Hence (super-)attracting cycles U_1, \dots, U_k that contain the critical point β may occur only if $\delta < |t| < 1$.

THEOREM 3. *For $0 < |t| < 1$, f_t is quasi-conformally conjugate to some polynomial*

$$P_c(z) = cz^m(z+1)^n \quad (c = c_t \neq 0)$$

with free critical point $-\frac{m}{m+n}$. The basin \mathcal{A}_t is completely invariant, and simply connected if and only if $t \notin \Omega_{\text{res}}^\alpha$. For $t \notin \Omega_0^\beta$, the Fatou set consists of \mathcal{A}_t , the simply connected basin \mathcal{B}_t and its pre-images and, additionally, of some (super-)attracting cycle of Fatou components and their pre-images if $t \in \mathbf{W}^\beta$; the cycle absorbs the critical point β .

Proof. To prove the second assertion we note that by Lemma 1 the pre-image D of the disc $\Delta = \Delta_{r|t|}$ is a simply connected Jordan domain that contains $\overline{\Delta} \cup [0, 1]$, but does not contain v_t^β . Then $D_2 = \widehat{\mathbb{C}} \setminus \overline{\Delta}$ is a backward invariant domain, and

$$f_t : D_1 \xrightarrow{d:1} D_2 \quad (D_1 = f_t^{-1}(D_2))$$

is a polynomial-like mapping in the sense of [6], of degree $d = m + n$, hence is hybrid equivalent to some polynomial P of degree d . We may assume that the quasi-conformal conjugation ψ_t with

$$\psi_t \circ f_t = P \circ \psi_t$$

maps $\infty, 0$, and -1 onto $0, \infty$, and -1 , respectively. Thus P is given by $P(z) = P_c(z) = cz^m(z+1)^n$, and ψ_t , hence also $c = c_t$ depends analytically on t . \square

REMARK. We note that $D_2 = D_2(|t|) = \{z : |z| > r|t|\}$ increases if $|t|$ decreases, while $D_1 = f_t^{-1}(\widehat{\mathbb{C}} \setminus \overline{\Delta}_{r|t|}) = f_1^{-1}(\widehat{\mathbb{C}} \setminus \overline{\Delta}_r)$ is independent of t . Thus the conformal modulus $\mu(|t|)$ of $D_2(|t|) \setminus \overline{D_1}$ satisfies

$\mu(1) \leq \mu(|t|) - \log \frac{1}{|t|} \leq \log \frac{\inf_{z \in D_1} |z|}{r}$. The bifurcation locus of P_c corresponds conformally to the bifurcation locus \mathbf{B}^β , and the hyperbolic components are just quasi-conformal images of the hyperbolic components of the quadratic family $z \mapsto z^2 + \xi$.

For $t \in \mathbf{W}_k$, the multiplier map $t \mapsto \lambda_t$ is an algebraic function of t . This is easily seen by writing the equations $f_t^k(z) = z$ and $\lambda = (f_t^k)'(z)$ as polynomial equations $q_1(z, t) = 0$ and $q_2(z, t, \lambda) = 0$, and computing the resultant $R_f(t, \lambda)$ of q_1 and q_2 with respect to z . For example, in case of $k = 1$, $m = 2$, and $n = 1$ we obtain

$$R_f(t, \lambda) = [-2 + 14t - 2t^2] + [1 - 10t + t^2]\lambda + 2t\lambda^2 = 0.$$

For $P_c(z) = cz^2(z + 1)$ we obtain in the same manner (multiplier μ)

$$R_P(c, \mu) = 9 + 2c - (c + 6)\mu + \mu^2 = 0.$$

Since the quasi-conformal conjugation respects multipliers ($\lambda_t = \mu_{c_t}$), c_t is an algebraic function of t by the identity theorem; in the present case we obtain $(1 + 2t + t^2 + 2tc)^2 = 0$ by computing the resultant of $R_f(t, \lambda)$ and $R_P(c, \lambda)$ with respect to λ , hence

$$t \mapsto c = c_t = -\frac{1}{2} \left(t + 2 + \frac{1}{t} \right) \quad (c = -\frac{9}{2} \leftrightarrow t = \frac{1}{2}(\sqrt{49} - \sqrt{45}))$$

maps $0 < |t| < 1$ conformally onto $\mathbb{C} \setminus [-2, 0]$.

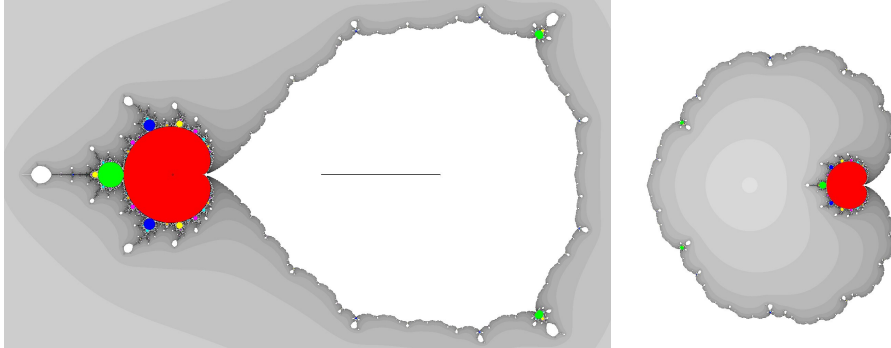


FIGURE 2-left: The parameter plane of $P_c(z) = cz^2(z + 1)$. The escape region for P_c (gray), the white region with slit, and the coloured domains correspond to $\Omega_{\text{res}}^\alpha$, $\Omega^\beta \cap \mathbb{D}$, and \mathbf{W}^β , respectively. The punctured disc $0 < |t| < 1$ corresponds to $\mathbb{C} \setminus [-2, 0]$ in the c -plane. FIGURE 2-right: The parameter plane of $P_{-\frac{1}{2}(t+2+\frac{1}{t})}$ in $-0.2 < \text{Re } t < 0.25$, $-0.25 < \text{Im } t < 0.25$ (see also Figure 1-right).

The following result was not explicitly stated but proved in [18]. The proof is an adaption of the procedure due to Douady [5], applied to the hyperbolic components of the quadratic family $R_t(z) = z^2 + t$ with one free critical point. The occurrence of several critical points requires a slightly more sophisticated argument. The present version applies

to a wider class of functions like $R_t(z) = z^d + t$, $R_t(z) = z^m + t/z^n$, $R_t(z) = t(1 + \frac{((d-1)^{d-1}/d^d)z^d}{1-z})$ ($d \geq 3$), $R_t(z) = -\frac{t(z^2-2)^2}{4z^2-1}$, the present family, and many others, to show that the hyperbolic components are simply connected and are mapped properly onto the unit disc by the multiplier map $t \mapsto \lambda_t$.

THEOREM 4. *Let $(R_t)_{t \in T}$ be any family of rational maps that is analytically parametrised over some domain T . Suppose that each R_t has a (super-)attracting cycle $U_0 \xrightarrow{m_1:1} U_1 \xrightarrow{m_2:1} \dots \xrightarrow{m_{n-1}:1} U_{n-1} \xrightarrow{m_n:1} U_n = U_0$, such that R_t^n has a single critical point $c_t \in U_0$ of multiplicity $m-1$, where $m = m_1 \dots m_n$ is the degree of $R_t^n : U_0 \xrightarrow{m:1} U_0$. Assume also that the multiplier λ_t satisfies $|\lambda_t| \rightarrow 1$ as $t \rightarrow \partial T$. Then the multiplier map $t \mapsto \lambda_t$ provides a proper map $T \xrightarrow{(m-1):1} \mathbb{D}$ which is ramified just over $w = 0$, and T is simply connected.*

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